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# The area-angular momentum inequality for black holes in cosmological spacetimes

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Received 2 February 2015

Accepted for publication 24 April 2015

Published 29 June 2015



CrossMark

## Abstract

For a stable, marginally outer trapped surface (MOTS) in an axially symmetric spacetime with cosmological constant  $\Lambda > 0$  and with matter satisfying the dominant energy condition, we prove that the area  $A$  and the angular momentum  $J$  satisfy the inequality  $8\pi|J| \leq A\sqrt{(1 - \Lambda A/4\pi)(1 - \Lambda A/12\pi)}$ , which is saturated precisely for the extreme Kerr–de Sitter family of metrics. This result entails a universal upper bound  $|J| \leq J_{\max} \approx 0.17/\Lambda$  for such MOTS, which is saturated for one particular extreme configuration. Our result sharpens the inequality  $8\pi|J| \leq A$  (Dain and Reiris 2011 *Phys. Rev. Lett.* **107** 051101, Jaramillo, Reiris and Dain 2011 *Phys. Rev. Lett.* D **84** 121503), and we follow the overall strategy of its proof in the sense that we first estimate the area from below in terms of the energy corresponding to a ‘mass functional’, which is basically a suitably regularized harmonic map  $\mathbb{S}^2 \rightarrow \mathbb{H}^2$ . However, in the cosmological case this mass functional acquires an additional potential term which itself depends on the area. To estimate the corresponding energy in terms of the angular momentum and the cosmological constant we use a subtle scaling argument, a generalized ‘Carter-identity’, and various techniques from variational calculus, including the mountain pass theorem.

Keywords: area inequality, apparent horizon, cosmological constant



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## 1. Introduction

Some remarkable area inequalities for stable marginally outer trapped surfaces (MOTS) have been proven recently [6–10, 14–16, 18–20]. In particular, for axially symmetric configurations with area  $A$  and angular momentum  $J$ , there is the bound [7, 14]

$$|J| \leq \frac{A}{8\pi}, \quad (1.1)$$

which is saturated for extreme Kerr black holes. Although a cosmological constant  $\Lambda$  does not explicitly enter into (1.1), this inequality holds in the presence of a non-negative  $\Lambda$ . On the other hand, when  $\Lambda > 0$ , stable MOTS obey the lower bound

$$A \leq 4\pi\Lambda^{-1}, \quad (1.2)$$

saturated for the extreme Schwarzschild–de Sitter horizon [12]. This readily implies the universal upper bound

$$|J| \leq (2\Lambda)^{-1} \quad (1.3)$$

which, however, can never be saturated even in theory (leaving practical considerations aside in view of the fact that  $\Lambda^{-1}$  is of order  $10^{122}$ ).

The situation bears some analogy to stable MOTS in (not necessarily axially symmetric) spacetimes with electromagnetic fields and electric and magnetic charges  $Q_E$  and  $Q_M$ . In this case the inequalities  $A \geq 4\pi Q^2$  [9] with  $Q^2 = Q_E^2 + Q_M^2$  (saturated for extreme Reissner–Nordström horizons) and  $A \leq 4\pi\Lambda^{-1}$  imply the (unsaturated) bound  $Q^2 \leq \Lambda^{-1}$ . There is however the stronger bound [16]

$$\Lambda A^2 - 4\pi A + 16\pi^2 Q^2 \leq 0 \quad (1.4)$$

which is saturated for extreme Reissner–Nordström–de Sitter configurations and, moreover, improves the universal charge bound to  $Q^2 \leq (4\Lambda)^{-1}$ .

Returning to the present axially symmetric case, the main objective of this article is to incorporate explicitly the cosmological constant into inequality (1.1) and determine how it controls the allowed values of the angular momentum. We prove the following theorem.

**Theorem 1.1.** *Let  $S$  be an axially symmetric, stable MOTS together with an axially symmetric 4-neighborhood of  $S$  called  $(\mathcal{N}, g_{ij})$ . On  $(\mathcal{N}, g_{ij})$  we require Einstein’s equations to hold, with  $\Lambda > 0$  and with matter satisfying the dominant energy condition (DEC). Then the angular momentum  $J$  and the area  $A$  of  $S$  satisfy*

$$|J| \leq \frac{A}{8\pi} \sqrt{\left(1 - \frac{\Lambda A}{4\pi}\right)\left(1 - \frac{\Lambda A}{12\pi}\right)}, \quad (1.5)$$

$$|J| \leq J_{\max} = \frac{3\sqrt{2}}{8\Lambda\sqrt{3}} \left(1 - \frac{1}{\sqrt{3}}\right) \approx \frac{0.17}{\Lambda}. \quad (1.6)$$

Here (1.5) is saturated precisely for the 1-parameter family of extreme Kerr–de Sitter (KdS) horizons while the universal bound (1.6) is saturated for one particular such configuration.

The proof of this theorem will be sketched in section 4, while details are postponed to section 5. We discuss now its scope and the main differences, similarities and difficulties compared to the ones cited above.

As  $\Lambda > 0$ , the main inequality (1.5) is stronger than both (1.1) and (1.2); in particular it forbids the black hole to rotate as fast as its non-cosmological counterpart. Concerning the saturation of (1.5), we observe the same pattern as in the previous inequalities: the extreme solutions set a bound to the maximum values of charges and/or angular momentum. The non-vanishing cosmological constant does not change this property of extreme black holes.

Inequality (1.6) is obtained in a straightforward manner from (1.5) and makes use of an interesting feature of the extreme KdS family. Given  $\Lambda > 0$  there exists a maximum value for the angular momentum which is attained at a certain value of the area  $A$ . This property is not shared by extreme Kerr horizons ( $\Lambda = 0$ ), where the value of  $A$  determines the angular momentum as  $8\pi |J| = A$ . Note also that, as opposed to (1.3), (1.6) is sharp and improves the numerical factor from 0.5 to 0.17 approximately.

As stated in theorem 1.1, the inequality (1.5) holds between the area and angular momentum of stable MOTS's. Nevertheless, due to the analogy between stable MOTS and stable minimal surfaces in maximal slices, one can prove an analogous result for this type of surfaces as well (see [6] for a discussion of the similarities of these surfaces within the context of geometric inequalities).

Note that matter satisfying the DEC is allowed. The energy condition is required in order to dispose of the matter terms and to arrive at the 'clean' inequality (1.5) where matter does not appear explicitly. However, for electromagnetic fields we expect to obtain an inequality between area, angular momentum, electromagnetic charges  $Q_E$ ,  $Q_M$  and cosmological constant which should reduce to (1.5) for  $Q = 0$  and to (1.4) when  $J = 0$ . We discuss a corresponding conjecture in section 6.

We now comment on the proof theorem 1.1 which is not a straightforward generalization of previous results. To explain this we recall briefly the basic strategy of [7, 14] that leads to (1.1). Starting with the stability condition one obtains a lower bound for the area of the MOTS in terms of a 'mass functional'  $\mathcal{M}$ . This  $\mathcal{M}$  is the key quantity in the proof, and depends only on the twist potential and the norm of the axial Killing vector. The non-negative cosmological constant and the matter terms (satisfying the DEC) neither appear in  $\mathcal{M}$  nor later in the discussion in this case. Therefore, the problem reduces to vacuum and with  $\Lambda = 0$ . Then, a variational principle is used to obtain a lower bound for  $\mathcal{M}$ . The key point in this step is the relation between  $\mathcal{M}$  and the 'harmonic energy' of maps between the two-sphere and the hyperbolic plane. This allows to use and adapt a powerful theorem by Hildebrandt *et al* [13] on harmonic maps, which gives existence and uniqueness of the minimizer for  $\mathcal{M}$ . This minimizer, in turn, gives the right hand side of (1.1).

In the present work where we strengthen (1.1) to (1.5), two important obstacles appear. Firstly, the area  $A$  now appears not only as upper bound on the corresponding functional  $\mathcal{M}$  but also explicitly in  $\mathcal{M}$  itself. This makes the variational principle hard to formulate. We overcome this problem in essence by 'freezing'  $A$  as well as  $J$  to certain values corresponding to an extreme KdS configuration, and by adapting the dynamical variables in  $\mathcal{M}$  suitably. Secondly, the relation of  $\mathcal{M}$  to harmonic maps mentioned above no longer persists, whence the proof of existence and uniqueness of a minimizer for  $\mathcal{M}$  has to be done here from scratch. We proceed by proving first that every critical point of  $\mathcal{M}$  is a local minimum. Finally we use the mountain pass theorem in order to get the corresponding global statement. Our paper is organized as follows.

In section 2 we recall and adapt some preliminary material, in particular the definition of angular momentum for general 2-surfaces, as well as the definition of a stable MOTS. In section 3 we discuss relevant aspects of the KdS metric, focusing on the extreme case. In

section 4 we sketch the proof of theorem 1.1, postponing the core of the argument to three key propositions which are proven in section 5.

In section 6 we conjecture a generalization of our inequality to the case with electromagnetic field along the lines mentioned above already, and we also discuss briefly the case  $\Lambda < 0$ .

## 2. Preliminaries

### 2.1. The geometric setup

We consider a manifold  $\mathcal{N}$  which is topologically a 4-neighborhood of an embedded 2-surface  $S$  of spherical topology.  $\mathcal{N}$  carries a metric  $g_{ij}$  and a Levi-Civita connection  $\nabla_i$ . (Latin indices from  $i$  onwards run from 0 to 3, and the metric has signature  $(-, +, +, +)$ ). The field equations are

$$G_{ij} = -\Lambda g_{ij} + 8\pi T_{ij}, \quad (2.1)$$

where  $\Lambda$  is the cosmological constant, and the energy momentum tensor  $T_{ij}$  satisfies the DEC. In sections 2 and 3 we allow  $\Lambda$  to have either sign; this enables us to compare with and to carry over useful formulas from work which focuses on Kerr–anti-de Sitter, in particular [4] and [5].

We next introduce null vectors  $\ell^i$  and  $k^i$  spanning the normal plane to  $S$  and normalized as  $\ell^i k_i = -1$ . We denote by  $q_{ij} = g_{ij} + 2\ell_{(i} k_{j)}$  the induced metric on  $S$ , the corresponding Levi-Civita connection by  $D_i$  and the Ricci scalar by  ${}^2R$ .  $\epsilon_{ij}$  and  $dS$  are respectively the volume element and the area measure on  $S$ . The normalization  $\ell_i k^i = -1$  leaves a (boost) rescaling freedom  $\ell'^i = f\ell^i$ ,  $k'^i = f^{-1}k^i$ . While this rescaling affects some quantities introduced below in an obvious way, our key definitions such as the angular momentum (2.4) and the definition of stability (2.12) are invariant, and the same applies to all our results. The expansion  $\theta^{(\ell)}$ , the shear  $\sigma_{ij}^{(\ell)}$  and the normal fundamental form  $\Omega_i^{(\ell)}$  associated with the null normal  $\ell^i$  are given by

$$\theta^{(\ell)} = q^{ij} \nabla_i \ell_j, \quad \sigma_{ij}^{(\ell)} = q^k{}_i q^l{}_j \nabla_k \ell_l - \frac{1}{2} \theta^{(\ell)} q_{ij}, \quad \Omega_i^{(\ell)} = -k^j q^k{}_i \nabla_k \ell_j. \quad (2.2)$$

### 2.2. Twist and angular momentum

We now assume that  $S$  as well as  $\Omega_i^{(\ell)}$  are axially symmetric, i.e. there is a Killing vector  $\eta^i$  on  $S$  such that

$$\mathcal{L}_\eta q_{ij} = 0 \quad \mathcal{L}_\eta \Omega_i^{(\ell)} = 0. \quad (2.3)$$

The field  $\eta^i$  is normalized so that its integral curves have length  $2\pi$ .

We define the angular momentum of  $S$  as

$$J = \frac{1}{8\pi} \int_S \Omega_i^{(\ell)} \eta^i dS, \quad (2.4)$$

which will be related to the Komar angular momentum shortly.

By Hodge's theorem, there exist scalar fields  $\omega$  and  $\lambda$  on  $S$ , defined up to constants, such that  $\Omega_i^{(\ell)}$  has the following decomposition

$$\Omega_i^{(\ell)} = \frac{1}{2\eta} \epsilon_{ij} D^j \omega + D_j \lambda. \quad (2.5)$$

From axial symmetry it follows that

$$\eta^i \Omega_i^{(\ell)} = \frac{1}{2\eta} \epsilon_{ij} \eta^i D^j \omega = \frac{1}{2} \eta^{-1/2} \xi^i D_i \omega, \quad (2.6)$$

where  $\eta = \eta^i \eta_i$  and  $\xi^i$  is a unit vector tangent to  $S$  and orthogonal to  $\eta^i$ .

We now recall from [7] that on any axially symmetric 2-surface one can introduce a coordinate system such that

$$q_{ij} dx^i dx^j = e^{2c} e^{-\sigma} d\theta^2 + e^\sigma \sin^2 \theta d\varphi^2 \quad (2.7)$$

for some function  $\sigma$  and a constant  $c$  which is related to the area  $A$  of  $S$  via  $A = 4\pi e^c$ . In such a coordinate system we can write  $J$  as

$$J = -\frac{1}{8} \int_0^\pi \omega' d\theta = -\frac{1}{8} [\omega(\pi) - \omega(0)], \quad (2.8)$$

where here and henceforth a prime denotes the derivative w.r.t.  $\theta$ . From now onwards we assume that the Killing vector  $\eta^i$  on  $S$  extends to  $\mathcal{N}$  as a Killing vector of  $g_{ij}$ . Of course this implies (2.3). Moreover, it follows that  $\mathcal{L}_\eta \ell^i = \mathcal{L}_\eta k^i = 0$ . Using the first equation we obtain

$$\eta^i \Omega_i^{(\ell)} = -k^j \ell^i \nabla_i \eta_j. \quad (2.9)$$

Inserting (2.9) in (2.4) we see that it indeed coincides with the Komar angular momentum

$$J = \frac{1}{8\pi} \int_S \nabla^i \eta^j dS_{ij}. \quad (2.10)$$

We finally introduce the twist vector

$$\omega_i = \epsilon_{ijkl} \eta^j \nabla^k \eta^l. \quad (2.11)$$

If the energy momentum tensor vanishes on  $\mathcal{N}$ , we have  $\nabla_{[i} \omega_{j]} = 0$ . Hence there exists a twist potential  $\omega$ , defined up to a constant, such that  $\omega_i = \nabla_i \omega$ . The restriction of this scalar field to  $S$  is easily seen to coincide with the  $\omega$  introduced in (2.5), which justifies the notation.

In what follows we will refer to the pair  $(\sigma, \omega)$  on  $S$  as the *data*.

### 2.3. Stable MOTS

We now take  $S$  to be a marginally trapped surface defined by  $\theta^{(\ell)} = 0$ . We will refer to  $\ell^i$  as the *outgoing* null vector, which leads to the name MOTS.

Moreover, following [2] (section 5) we now consider a family of two-surfaces in a neighborhood of  $S$  together with respective null normals  $\ell_i$  and  $k_i$  and we impose the following additional requirements on  $S$  and its neighborhood.

**Definition 2.1.** A marginally trapped surface  $S$  is stable if there exists an outgoing ( $-k^i$ -oriented) vector  $X^i = \gamma \ell^i - \psi k^i$ , with  $\gamma \geq 0$  and  $\psi > 0$ , such that the variation  $\delta_X$  of  $\theta^{(\ell)}$  with respect to  $X^i$  fulfills the condition

$$\delta_X \theta^{(\ell)} \geq 0. \quad (2.12)$$

Two remarks are in order here. Firstly, it is easy to see (see section 5 of [2]) that stability of  $S$  w.r.t. some direction  $X^i$  implies stability w.r.t all directions ‘tilted away from’  $\ell^i$ . In particular, since  $\delta_{-\psi k} \theta^{(\ell)} \geq \delta_X \theta^{(\ell)}$  stability w.r.t. any  $X^i$  implies stability in the

past outgoing null direction  $-k^i$ . This latter condition suffices as requirement for all our results.

The other remark concerns the relation between stability and axial symmetry. We recall that in [7, 14], inequality (1.1) was proven under the symmetry requirements (2.3) and under a stability condition similar to definition 2.1 which, however, required  $\psi$  to be axially symmetric as well. (Axial symmetry of  $\gamma$  was also assumed but not used in the proof). In contrast, in the present theorem (1.1) we impose the stronger symmetry requirement that  $\mathcal{S}$  as well as its neighborhood  $\mathcal{N}$  are axially symmetric. In this case it suffices to impose the stability condition (2.12) as above, namely without explicitly requiring axial symmetry of  $\psi$ , since the existence of an axially symmetric function  $\tilde{\psi}$  then follows automatically, see theorem 8.2. of [2]. Moreover, for *strictly* stable MOTS (which satisfy  $\delta_\chi \theta^{(\ell)} \neq 0$  in addition to (2.12)) there follows even axial symmetry of the surface itself if its neighborhood is axially symmetric (see theorem 8.1. of [2]).

### 3. Kerr–de Sitter

In this section we review some relevant properties of the event horizons of the KdS solutions, making use of [4, 5], and references therein. Other aspects of the rich and complex structure of these spacetimes can be found in [11].

#### 3.1. The metric, the horizon and the angular momentum

In ‘Boyer–Lindquist’ coordinates, the KdS metric is

$$ds^2 = -\frac{\zeta}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\kappa} d\phi \right)^2 + \frac{\rho^2}{\zeta} dr^2 + \frac{\rho^2}{\chi} d\theta^2 + \frac{\chi \sin^2 \theta}{\rho^2} \left( a dt - \frac{r^2 + a^2}{\kappa} d\phi \right)^2, \quad (3.1)$$

where

$$\zeta = (r^2 + a^2) \left( 1 - \frac{\Lambda r^2}{3} \right) - 2mr, \quad \rho^2 = r^2 + a^2 \cos^2 \theta \quad (3.2)$$

$$\kappa = 1 + \frac{\Lambda a^2}{3}, \quad \chi = 1 + \frac{\Lambda a^2 \cos^2 \theta}{3}, \quad (3.3)$$

where  $m$  and  $a$  satisfy certain upper bounds, cf. [4, 5].

As a function of  $r$ ,  $\zeta$  has one negative root and three positive roots (possibly counted with multiplicities). The greatest root,  $r_{\text{ch}}$ , marks the cosmological horizon, while the second greatest,  $r_h$ , marks the event horizon (from now on simply called ‘horizon’).

The area of the horizon is

$$A = \frac{4\pi (r_h^2 + a^2)}{\kappa} \quad (3.4)$$

and the induced metric on it reads

$$ds^2 = \underbrace{\frac{\mu_h^2}{\kappa^2 \rho_h^2}}_{e^\sigma} \left( \underbrace{\frac{\kappa^2 \rho_h^4}{\mu_h^2 \chi}}_{e^{2q}} d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad (3.5)$$

where  $\mu_h^2 = (r_h^2 + a^2)^2 \chi$  and  $\rho_h = r_h^2 + a^2 \cos^2 \theta$ . Hence

$$e^{\sigma+q} = \frac{r_h^2 + a^2}{\kappa} = e^c = \text{const.} = \frac{A}{4\pi} \quad (3.6)$$

and the metric is in the ‘canonical form’ (2.7) of [7]

$$ds^2 = e^\sigma \left( e^{2q} d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (3.7)$$

with  $\sigma + q = c = \text{const.}$

We now calculate the twist potential  $\omega(\eta)$  everywhere (not only on  $S$ ), for  $\eta^a = \partial/d\phi$ . Adapting a known calculation in the case  $\Lambda = 0$  (see e.g. appendix A of [3]) and omitting some intermediate steps, we find

$$\omega' = \omega_\theta = \epsilon_{\theta\phi r t} g^{rr} g^{tt} \partial_r \eta_t + \epsilon_{\theta\phi r t} g^{rr} g^{t\phi} \partial_r \eta_\phi = -\frac{\zeta \sin \theta}{\kappa} \left( g^{tt} \partial_r g_{t\phi} + g^{t\phi} \partial_r g_{\phi\phi} \right) \quad (3.8)$$

$$= -\frac{\kappa}{\chi \sin \theta} \left( g_{\phi\phi} \partial_r g_{t\phi} - g_{t\phi} \partial_r g_{\phi\phi} \right) = -\frac{2ma \sin^3 \theta}{\kappa^2 \rho^2} \left[ r^2 - a^2 + \frac{2r^2}{\rho^2} (r^2 + a^2) \right] \quad (3.9)$$

$$= -\frac{2ma}{\kappa^2} \frac{\partial}{\partial \theta} \left( \cos^3 \theta - 3 \cos \theta - \frac{a^2 \cos \theta \sin^4 \theta}{\rho^2} \right). \quad (3.10)$$

It follows that

$$\omega = -\frac{2ma}{\kappa^2} \left( \cos^3 \theta - 3 \cos \theta - \frac{a^2 \cos \theta \sin^4 \theta}{\rho^2} \right). \quad (3.11)$$

We note that compared to the case  $\Lambda = 0$ ,  $\omega$  just gets an extra factor  $1/\kappa^2$ . Integrating and using (2.8) we obtain in particular that

$$J = am/\kappa^2 \quad (3.12)$$

which agrees with equation (2.10) of [5] and equation (18) of [4].

### 3.2. Extreme horizons

When at least two of the three non-negative roots of  $\zeta(r)$  coincide, (one of which is necessarily  $r_h$ ), the horizon is called extremal. When this happens the geometry near the horizon degenerates to a ‘throat’. We refer to [5] for a further discussion. In what follows we will just need the relation between the parameters  $m, a, \Lambda, A$  and  $J$  which we derive explicitly.

For extremal event horizons the radius of the limiting sphere  $r_e$  satisfies, in addition to  $\zeta(r_e) = 0$ , the equation



$$0 = \frac{1}{2} \frac{d\zeta}{dr} \Big|_e = -\frac{2\Lambda r_e^3}{3} + r_e \left( 1 - \frac{\Lambda a^2}{3} \right) - m. \quad (3.13)$$

Here and henceforth a subscript  $e$  indicates extremality. Eliminating  $m$  from  $\zeta(r_e) = 0$  and (3.13) we obtain

$$\Lambda r_e^4 + r_e^2 \left( \frac{\Lambda a^2}{3} - 1 \right) + a^2 = 0. \quad (3.14)$$

For  $\Lambda \leq 0$  this equation has just a single root which can be called extremal horizon, while for  $\Lambda > 0$  there are two solutions  $r_e = r_{\pm}$  for given  $J$ . Explicitly, for  $\Lambda > 0$ ,

$$r_{\pm}^2 = \frac{1}{2\Lambda} \left( 1 - \frac{\Lambda a^2}{3} \right) \pm \frac{1}{2\Lambda} \sqrt{\left( 1 - \frac{\Lambda a^2}{3} \right)^2 - 4a^2\Lambda}. \quad (3.15)$$

When  $r_e = r_-$ , (and  $r_e$  is not a triple root), the first two positive roots meet and  $r_e < r_{ch}$ , which means that a cosmological horizon persists in spacetime. On the other hand when  $r_e = r_+$ , then the last two positive roots coincide and the event and the cosmological horizons become both extremal (and merge).

Using (3.14) to eliminate  $a^2$  from (3.4) we find

$$A = \frac{8\pi r_e^2}{1 + \Lambda r_e^2}. \quad (3.16)$$

On the other hand, eliminating  $r_e$  from (3.14) and (3.4) gives

$$a^2 = \frac{A}{4\pi} \frac{1 - \Lambda A/4\pi}{(1 - \Lambda A/8\pi)(1 - \Lambda A/12\pi)}. \quad (3.17)$$

In equation (3.12) we eliminate now  $m$  using (3.13), then  $a^2$  using (3.14) and finally  $r_e^2$  using (3.16). We obtain the following simple relation between the angular momentum and the area for extreme K(a)dS

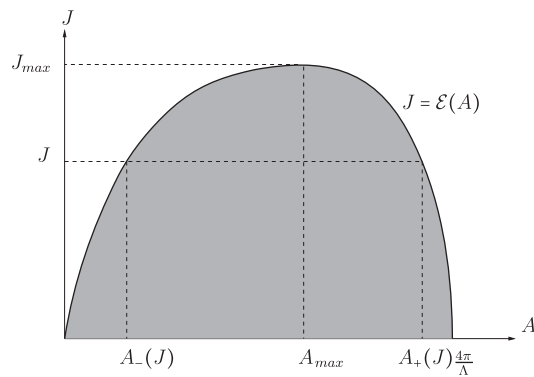
$$|J| = \mathcal{E}(A) := \frac{A}{8\pi} \sqrt{\left( 1 - \frac{\Lambda A}{4\pi} \right) \left( 1 - \frac{\Lambda A}{12\pi} \right)} \quad (3.18)$$

which after a trivial reformulation agrees with (2.32) of [5]. In the case  $\Lambda > 0$  and  $J = 0$  the zeros of the parentheses correspond to the black hole horizon and the cosmological horizon of Schwarzschild–de Sitter, respectively.

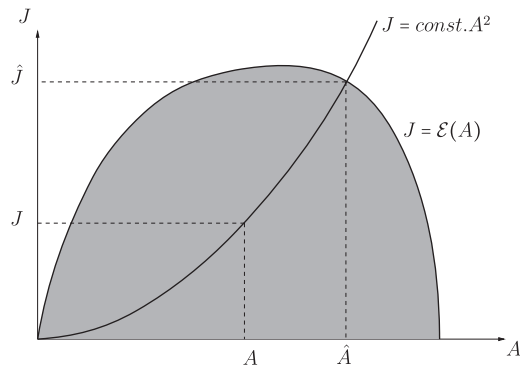
For  $\Lambda > 0$  we are only interested in the domain  $\Lambda A/4\pi < 1$ —recall that this bound can be shown for *all* stable MOTS (irrespectively of spherical symmetry) [12]. In this range of  $A$ , (3.18) takes on a maximal value

$$J_{\max} = \frac{3\sqrt{2}}{8\Lambda\sqrt{3}} \left( 1 - \frac{1}{\sqrt{3}} \right) \approx \frac{0.17}{\Lambda} \quad \text{at} \quad A_{\max} = \frac{6\pi}{\Lambda} \left( 1 - \frac{1}{\sqrt{3}} \right) \quad (3.19)$$

which is the value stated in (1.6). Moreover, for each  $J$  with  $|J| < J_{\max}$  there are *two* values  $A_-(J) < A_+(J)$  for the area, see figure 1. We are now ready to describe the proof of theorem 1.1.



**Figure 1.** The shaded region represents all points satisfying  $|J| \leq \mathcal{E}(A)$ .



**Figure 2.** The construction described in theorem 4.1.

#### 4. The structure and the proof of the main theorem

The main inequality

$$|J| \leq \mathcal{E}(A) \tag{4.1}$$

with  $\mathcal{E}$  given in (3.18) and  $\Lambda > 0$  will not be shown directly but it will follow from a related one. This is explained in the following theorem:

**Theorem 4.1.** *For any given MOTS with area  $A$ , cosmological constant  $\Lambda$  and angular momentum  $J$ , there is a unique extreme KdS solution with area  $\hat{A}$  constant  $\Lambda$  and angular momentum  $\hat{J}$  such that*

$$\frac{|J|}{A^2} = \frac{|\hat{J}|}{\hat{A}^2}, \tag{4.2}$$

and  $\hat{A}\Lambda \leq 4\pi$ . Moreover, the inequality  $|J| \leq \mathcal{E}(A)$  is equivalent to the inequality

$$\hat{A} \geq A. \tag{4.3}$$

**Proof.** The first result, leading to equation (4.2), is intuitively clear from figure 2 since through any point  $(A, J)$  there is a unique parabola  $J/A^2 = \text{const.}$ , and any such parabola intersects the ‘extreme’ curve  $J = \mathcal{E}(A)$  precisely once apart from the trivial point  $(0, 0)$ . To state this rigorously, let  $\lambda := A/\hat{A}$  and hence  $|\hat{J}| = \lambda^2 |J|$  and  $\hat{A}\Lambda \leq 4\pi$ . Then the hatted version of (3.18) gives a quadratic equation for  $\lambda(J, A)$ . If  $32\pi^2\sqrt{3} |J| > \Lambda A^2$  this equation has a unique solution other than  $(0, 0)$ . Otherwise, there are two non-trivial solutions but only one of them lies in the region of interest  $\hat{A}\Lambda \leq 4\pi$ .

To prove the equivalence between (4.1) and (4.3), assume first that  $\hat{A} \geq A$ . Then

$$A^2 \leq \hat{A}^2 = |\hat{J}| \frac{A^2}{|J|} = \mathcal{E}(\hat{A}) \frac{A^2}{|J|} = \frac{\mathcal{E}(\lambda A) \hat{A}^2}{\lambda^2 |J|}, \quad (4.4)$$

where we have used (4.2), (3.18) and  $\hat{A} = \lambda A$ , respectively. We next use that the function  $\frac{\mathcal{E}(\lambda A)}{\lambda^2}$  is monotonically decreasing with  $\lambda$  and therefore, as  $\hat{A} \geq A$  we bound the last term as  $\frac{\mathcal{E}(\lambda A)}{\lambda^2} \leq \mathcal{E}(A)$ . Putting this together with (4.4) we find

$$\hat{A}^2 \leq \mathcal{E}(A) \frac{\hat{A}^2}{|J|} \quad (4.5)$$

which gives the desired result, that is, that (4.3) implies (4.1).

To prove the converse assume  $|J| \leq \mathcal{E}(A)$ . Then  $\hat{J} = \lambda^2 J$  and (3.18) give

$$\mathcal{E}(\lambda A) = |\hat{J}| = \lambda^2 |J| \leq \lambda^2 \mathcal{E}(A) \quad (4.6)$$

and therefore

$$\frac{\mathcal{E}(\lambda A)}{\lambda^2} \leq \mathcal{E}(A). \quad (4.7)$$

Again, due to the monotonicity of the left hand side with respect to  $\lambda$  we obtain  $\lambda A \geq A$  which is (4.3).  $\square$

Having established the equivalence between the main inequality (4.1) and (4.3), the next section will be devoted to proving (4.3) for a stable MOTS  $\mathcal{S}$  with area  $A$ , angular momentum  $J$  and data  $(\sigma, \omega)$ . The proof consists of the same two steps as in the case  $\Lambda = 0$ . However, as we mentioned in the introduction and as we will see below, when  $\Lambda > 0$  many new complications arise.

In general terms the basic steps can be described as follows.

**Step I.** We write the stability inequality (2.12) in terms of the data  $(\sigma, \omega)$  and multiply it by an axially symmetric function  $\alpha^2$  whose choice is motivated by the form of the data  $(\sigma, \omega)$  of the extreme KdS horizon. Then we integrate it on  $\mathcal{S}$  to obtain a lower bound for  $A$  in terms of the so-called mass functional  $\mathcal{M}$  depending on the dynamic variables  $(\sigma, \omega)$ . The result is the following proposition:

**Proposition 4.2.** *Let  $(\sigma, \omega)$  be the data of a stable MOTS of area  $A$  and angular momentum  $J$ . Then, for any real number  $a$  the following inequality holds*

$$\frac{A}{4\pi} \geq e^{\frac{\mathcal{M}(\sigma, \omega, A, a) - \beta}{8\kappa}}, \quad (4.8)$$

where the functional  $\mathcal{M}$  is given by

$$\mathcal{M}(\sigma, \omega, A, a) := \int_0^\pi \left[ \sigma'^2 + \frac{\omega'^2}{\eta^2} + 4\sigma \frac{(1 + \Lambda a^2 \cos^2 \theta)}{\chi} + 4 \left( \frac{A}{4\pi} \right)^2 \Lambda e^{-\sigma} \right] \chi \sin \theta d\theta, \quad (4.9)$$

where

$$\beta = \int_0^\pi \left( 4\chi + \frac{\chi'^2}{\chi} \right) \sin \theta d\theta \quad (4.10)$$

and where  $\chi(a)$  has been defined in (3.3).

At this stage the constant  $a$  is arbitrary, but it will be fixed in the next step.

Step II. The difficulty now is to choose  $a$  conveniently and to show that, with such  $a$ , the rhs of (4.8) has the lower bound  $A^2/4\pi\hat{A}$ . This would prove (4.3), (hence (4.1) by theorem 4.1).

We choose  $a$  equal to the value that it would take for the extreme black hole of area  $\hat{A}$ . The explicit form is (3.17) with  $A$  replaced by  $\hat{A}$ . We will denote it by  $\hat{a}$  and we denote by  $\hat{\kappa}$ ,  $\hat{\chi}$  and  $\hat{\beta}$ , the values of  $\kappa$ ,  $\chi$  and  $\beta$  when  $a$  is replaced by  $\hat{a}$  in (3.3), and (4.10). Then, for the data  $(\sigma, \omega)$  of the given MOTS define

$$\hat{\sigma} := \sigma + 2 \ln \lambda \quad \hat{\omega} = \lambda^2 \omega, \quad (4.11)$$

where (again)  $\lambda = A/\hat{A}$ . With this change of variables we obtain

$$\mathcal{M}(\sigma, \omega, A, \hat{a}) = \mathcal{M}(\hat{\sigma}, \hat{\omega}, \hat{A}, \hat{a}) - 16\kappa \ln(\hat{A}/A). \quad (4.12)$$

Thus

$$\frac{A}{4\pi} \geq e^{\frac{\mathcal{M}(\sigma, \omega, A, \hat{a}) - \hat{\beta}}{8\hat{\kappa}}} = \left( \frac{A}{\hat{A}} \right)^2 e^{\frac{\mathcal{M}(\hat{\sigma}, \hat{\omega}, \hat{A}, \hat{a}) - \hat{\beta}}{8\hat{\kappa}}} \quad (4.13)$$

and we need to prove proposition 4.3.

**Proposition 4.3.** *In the setup explained above we have*

$$e^{\frac{\mathcal{M}(\hat{\sigma}, \hat{\omega}, \hat{A}, \hat{a}) - \hat{\beta}}{8\hat{\kappa}}} \geq \frac{\hat{A}}{4\pi}. \quad (4.14)$$

We wish to mention the following point here. (4.14) means that the lower bound is obtained by minimizing the functional  $\mathcal{M}(\hat{\sigma}, \hat{\omega}, \hat{A}, \hat{a})$  among all pairs  $(\hat{\sigma}, \hat{\omega})$  of smooth functions with  $8\pi\hat{J} = -(\hat{\omega}(\pi) - \hat{\omega}(0))$ . A particular class of such functions has been constructed above via (4.11) from smooth data  $(\sigma, \omega)$  on a smooth MOTS of area  $A$  and angular momentum  $J$ . However, this does *not* mean that  $(\hat{\sigma}, \hat{\omega})$  will still form smooth data on a smooth MOTS of area  $\hat{A}$  and angular momentum  $\hat{J}$ . This can be seen as follows. In order for the MOTS to be smooth (free of conical singularities), the coordinate function  $q$  must vanish at the poles, i.e.  $q(0) = q(\pi) = 0$  which implies that  $A = 4\pi e^{\sigma(0)} = 4\pi e^{\sigma(\pi)}$ . But inserting the scaling law (4.11) in the latter relation contradicts the smoothness property  $\hat{A} = 4\pi e^{\hat{\sigma}(0)} = 4\pi e^{\hat{\sigma}(\pi)}$  for the hatted data, (except in the trivial case  $\lambda = 1$ ). Therefore,  $\mathcal{M}(\hat{\sigma}, \hat{\omega}, \hat{A}, \hat{a})$  should be considered as ‘abstract’ functional in the sense that its arguments are no longer directly related to any

MOTS. Nevertheless, extreme KdS is not only a critical point of  $\mathcal{M}(\sigma, \omega, A, a)$  but also of  $\mathcal{M}(\hat{\sigma}, \hat{\omega}, \hat{A}, \hat{a})$ , and the properties of the latter functional enable us to prove (4.14).

Next we present the proofs of propositions 4.2 and 4.3.

## 5. Proof of the main propositions

### 5.1. Proof of proposition 4.2

**Proof.** The proof is analogous to the case  $\Lambda = 0$  [14] to which it reduces by setting  $\chi \equiv 1$ . The starting point is the stability inequality (2.12) in which we take  $\psi$  to be axially symmetric without loss of generality (see the remarks after definition 2.1). In terms of the quantities introduced in section 2 we obtain, integrating (2.12) against any axisymmetric function  $\alpha: S \rightarrow \mathbb{R}$ ,

$$\int_S \left( |D\alpha|^2 + \frac{2R}{2}\alpha^2 - \alpha^2 |\Omega^{(\ell)}|^2 - \Lambda\alpha^2 \right) dS \geq 0. \quad (5.1)$$

As mentioned in the previous section, we choose the trial function based on the form of the extreme KdS geometry as

$$\alpha = \chi^{1/2} e^{-\sigma/2}. \quad (5.2)$$

In the coordinates (3.7) the scalar curvature takes the form

$$2R = \frac{e^{\sigma-2c}}{\sin \theta} \left[ -2\sigma' \cos \theta - \sin \theta \sigma'^2 + 2 \sin \theta - (\sin \theta \sigma')' \right]. \quad (5.3)$$

Using this expression we obtain

$$\frac{1}{2\pi} \int_S \left( |D\alpha|^2 + \frac{2R}{2}\alpha^2 \right) dS = \int_0^\pi \left( \frac{\chi \sigma'^2}{4} - \frac{\sigma' \chi'}{2} + \frac{\chi'^2}{4\chi} \right) \sin \theta d\theta \quad (5.4)$$

$$\int_0^\pi \left( -\chi \sigma' \cos \theta - \frac{\chi \sigma'^2 \sin \theta}{2} + \chi \sin \theta - \frac{\chi (\sin \theta \sigma')'}{2} \right) d\theta. \quad (5.5)$$

Integration by parts and some rearrangement yields

$$\frac{1}{2\pi} \int_S \left( |D\alpha|^2 + \frac{2R}{2}\alpha^2 \right) dS = - \int_0^\pi \left[ \frac{\sigma'^2}{4} + \sigma \left( 1 + \frac{2\Lambda a^2}{3} \cos \theta \right) \right] \sin \theta d\theta \quad (5.6)$$

$$+ \int_0^\pi \left( \chi + \frac{\chi'^2}{4\chi} \right) \sin \theta d\theta - \chi \sigma \cos \theta \Big|_0^\pi. \quad (5.7)$$

Using (3.6), the last term in line (5.7) above is equal to  $2\kappa \ln(A/4\pi)$ . Finally, still following [14], we have

$$-\frac{1}{2\pi} \int_S \left( \alpha^2 |\Omega^{(\ell)}|^2 + \Lambda\alpha^2 \right) dS = -\frac{1}{4} \int_0^\pi \chi \frac{\omega'^2}{e^{2\sigma} \sin^4 \theta} \sin \theta d\theta - \Lambda e^{2c} \int_0^\pi \chi e^{-\sigma} \sin \theta d\theta. \quad (5.8)$$

Combining equations (5.1), (5.6)–(5.7) and (5.9) we find

$$2\kappa \ln \left( \frac{A}{4\pi} \right) \geq \frac{\mathcal{M} - \beta}{4} \tag{5.9}$$

with  $\beta$  as in (4.10). This expression is equivalent to (4.8) as wished.  $\square$

5.2. Proof of proposition 4.3

In this section we prove (4.14) where the hatted variables ( $\hat{\sigma}$ ,  $\hat{\omega}$ ) refer to the rescaled quantities introduced in (4.11). To simplify the notation, for this section only, we omit the hats on these functions. With the new notation, inequality (4.14) reads

$$e^{\frac{\mathcal{M}(\sigma, \omega, \hat{A}, \hat{a}) - \beta}{8\hat{\kappa}}} \geq \frac{\hat{A}}{4\pi}. \tag{5.10}$$

As in the proof of the inequality in the  $\Lambda = 0$  case, this step is done by minimizing the functional  $\mathcal{M}$ . We find first a minimum of  $\mathcal{M}$  for functions  $\sigma$ ,  $\omega$  defined on compact intervals  $[\theta_a, \theta_b] \in (0, \pi)$  (in propositions 5.1 and 5.2), and then take the limit  $[\theta_a, \theta_b] \rightarrow [0, \pi]$  to find (5.10) (in proposition 5.3). Recall that when  $\Lambda = 0$  the extreme Kerr geometry is the minimizer of the corresponding functional.

In this  $\Lambda > 0$  case, we find by a straightforward computation that extreme KdS data  $(\sigma_e, \omega_e)$  is a critical point of  $\mathcal{M}$ , that is, the explicit functions

$$e^{\sigma_e} = \frac{\hat{\mu}_e^2}{\hat{\kappa}_e^2 \hat{\rho}_e^2}, \quad \omega_e' = \frac{-2\hat{\chi} \hat{a} \hat{r}_e (\hat{r}_e^2 + \hat{a}^2)^2 \sin^3 \theta}{\hat{\mu}_e \hat{\rho}_e^4} \tag{5.11}$$

satisfy the Euler–Lagrange equations of  $\mathcal{M}$ :

$$\frac{1}{\sin \theta} \frac{d}{d\theta} (2\hat{\chi} \partial_\theta \sigma \sin \theta) = -\frac{2\hat{\chi} \omega'^2}{\eta^2} + 4(1 + \Lambda \hat{a}^2 \cos^2 \theta) - \frac{\Lambda \hat{\chi} \hat{A}^2}{4\pi^2} e^{-\sigma} \tag{5.12}$$

$$\frac{d}{d\theta} \left( \sin \theta \frac{\hat{\chi} \partial_\theta \omega}{\eta^2} \right) = 0. \tag{5.13}$$

In (5.11), the quantities  $\hat{\rho}_e$ ,  $\hat{\kappa}_e$ ,  $\hat{\mu}_e$ ,  $\hat{r}_e$  and  $\hat{\chi}_e$  were defined in (3.2), (3.3), below (3.5) and in (3.14) but carrying subscripts and hats they refer here to the extreme KdS solution with parameter  $\hat{a}$ . Using (3.14) it is easy to see that the above  $\omega_e'$  indeed coincides with (3.9) and therefore with (3.10).

This property of extreme KdS geometry will play a fundamental role in the proof of (5.10), but before going into details, some preliminary definitions are needed.

**Preliminaries.** Let  $0 < \theta_a < \theta_b < \pi$  be fixed. For any function  $f: [\theta_a, \theta_b] \rightarrow \mathbb{R}$  in  $H^{1,2}$  define

$$\|f\|_2^2 := \|f\|_{L^2}^2 = \int_{\theta_a}^{\theta_b} f^2 \, d\theta, \tag{5.14}$$

$$\|f\|_{1,2}^2 := \|f\|_{H^{1,2}}^2 = \int_{\theta_a}^{\theta_b} \left[ (\partial_\theta f)^2 + f^2 \right] d\theta = \|\partial_\theta f\|_2^2 + \|f\|_2^2. \tag{5.15}$$

Then, for any  $\theta_1 < \theta_2$ , ( $\theta_a < \theta_1$  and  $\theta_2 < \theta_b$ ), we have

$$|f(\theta_1) - f(\theta_2)|^2 \leq |\theta_2 - \theta_1| \|f\|_{1,2}^2. \quad (5.16)$$

This says in particular that  $f$  is uniformly continuous and we have

$$\|f - f_a\|_\infty^2 := \sup \left\{ (f - f_a)^2(\theta) : \theta \in [\theta_a, \theta_b] \right\} \leq \pi \|\partial_\theta f\|_2^2 \leq \pi \|f\|_{1,2}^2, \quad (5.17)$$

where  $f_a = f(\theta_a)$ .

We will use the affine space  $\Gamma_{ab}$  of  $H^{1,2}$  paths  $\gamma: [\theta_a, \theta_b] \rightarrow \mathbb{R}^2$ ,  $\gamma = (\sigma, \omega)$ , such that  $(\sigma(\theta_a), \omega(\theta_a)) = (\sigma_e(\theta_a), \omega_e(\theta_a))$  and  $(\sigma(\theta_b), \omega(\theta_b)) = (\sigma_e(\theta_b), \omega_e(\theta_b))$ ,  $(5.18)$

where  $(\sigma_e, \omega_e)$  are the data of extreme KdS of area  $\hat{A}$ .

In line with the notation (5.14) we use the shorthand  $\|\gamma_1 - \gamma_2\|_{1,2}^2 := \|\sigma_1 - \sigma_2\|_{1,2}^2 + \|\omega_1 - \omega_2\|_{1,2}^2$ .

Let  $\mathcal{M}_{ab} = \mathcal{M}_{ab}(\gamma): \Gamma_{ab} \rightarrow \mathbb{R}$  be the functional given by

$$\mathcal{M}_{ab}(\gamma) = \int_{\theta_a}^{\theta_b} \left( (\partial_\theta \sigma)^2 + 4\sigma \frac{(1 + \Lambda \hat{a}^2 \cos^2 \theta)}{\hat{\chi}} + \frac{(\partial_\theta \omega)^2}{e^{2\sigma} \sin^4 \theta} + 4\Lambda \left( \frac{\hat{A}}{4\pi} \right)^2 e^{-\sigma} \right) \hat{\chi} \sin \theta d\theta. \quad (5.19)$$

Note that this functional is the same as the  $\mathcal{M}$  appearing in (5.10) except that the integration is over  $[\theta_a, \theta_b]$  and that the arguments  $\gamma = (\sigma, \omega)$  vary in  $\Gamma_{ab}$ .

**The functional  $\overline{\mathcal{M}}_{ab}$ .** Consider the change of variables  $(\theta, \sigma, \omega) \rightarrow (\bar{\theta}, \bar{\sigma}, \bar{\omega})$  given by

$$\frac{d\bar{\theta}}{d\theta} = \frac{\sin \bar{\theta}}{\sin \theta \hat{\chi}(\theta)}, \quad \bar{\sigma} = \sigma + 2 \ln \frac{\sin \theta}{\sin \bar{\theta}}, \quad \bar{\omega} = \omega. \quad (5.20)$$

Explicitly,  $\bar{\theta}(\theta)$  reads, with a suitable choice of the integration constant,

$$\tan \frac{\bar{\theta}}{2} = \left( \tan \frac{\theta}{2} \right)^{1/\hat{\kappa}} \exp \left[ -\frac{\hat{a}}{\hat{\kappa}} \sqrt{\frac{\Lambda}{3}} \arctan \left( \hat{a} \sqrt{\frac{\Lambda}{3}} \cos \theta \right) \right]. \quad (5.21)$$

It follows that the map  $\theta \rightarrow \bar{\theta}$  is a diffeomorphism from  $[0, \pi]$  into  $[0, \pi]$  and that  $0 < c_1 < (\sin \theta / \sin \bar{\theta}) < c_2 < \infty$  for  $c_1$  and  $c_2$  depending only on  $\hat{a}^2 \Lambda$ . The transformation of the affine space  $\Gamma_{ab}$  will be denoted by  $\overline{\Gamma}_{ab}$ . A straightforward computation shows

$$\begin{aligned} \mathcal{M}_{ab}(\gamma) &= \overline{\mathcal{M}}_{ab}(\bar{\gamma}) \\ &+ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \frac{4 \cos^2 \bar{\theta}}{\sin \bar{\theta}} d\bar{\theta} + 4\bar{\sigma} \cos \bar{\theta} \Big|_{\bar{\theta}_a}^{\bar{\theta}_b} 4\sigma \left( \cos \theta + \frac{\hat{a}^2 \Lambda}{3} \cos^3 \theta \right) \Big|_{\theta_a}^{\theta_b} \\ &- \int_{\theta_a}^{\theta_b} \frac{4\hat{\chi} \cos^2 \theta}{\sin \theta} d\theta, \end{aligned} \quad (5.22)$$

where the functional  $\overline{\mathcal{M}}_{ab} = \overline{\mathcal{M}}_{ab}(\bar{\gamma}): \overline{\Gamma}_{ab} \rightarrow \mathbb{R}$  is given by

$$\overline{\mathcal{M}}_{ab}(\bar{\gamma}) = \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( (\partial_{\bar{\theta}} \bar{\sigma})^2 + 4\bar{\sigma} + \frac{(\partial_{\bar{\theta}} \bar{\omega})^2}{e^{2\bar{\sigma}} \sin^4 \bar{\theta}} + \left[ 4\Lambda \left( \frac{\hat{A}}{4\pi} \right)^2 \frac{\sin^4 \theta}{\sin^4 \bar{\theta}} \hat{\chi}^2(\theta) \right] e^{-\bar{\sigma}} \right) \sin \bar{\theta} d\bar{\theta}. \quad (5.23)$$

Thus, the functionals  $\mathcal{M}_{ab}: \Gamma_{ab} \rightarrow \mathbb{R}$  and  $\overline{\mathcal{M}}_{ab}: \overline{\Gamma}_{ab} \rightarrow \mathbb{R}$  differ by a constant and boundary terms. This immediately implies that  $\gamma$  is a critical point of  $\mathcal{M}_{ab}$  iff  $\bar{\gamma}$  is a critical point of  $\overline{\mathcal{M}}_{ab}$ . In particular as  $\gamma_e$  is a critical point of  $\mathcal{M}_{ab}$ ,  $\bar{\gamma}_e$  is a critical point of  $\overline{\mathcal{M}}_{ab}$ . As we will explain below, the nature of critical points of the functional  $\overline{\mathcal{M}}_{ab}$  can be easily analysed via a crucial formula due to Carter. A similar simple formula to analyse the critical points of  $\mathcal{M}_{ab}$  is unknown to us. For this reason we will continue working with  $\overline{\mathcal{M}}_{ab}$  rather than with  $\mathcal{M}_{ab}$ .

**The results.** The next three propositions together prove proposition 4.3. Propositions 5.1 and 5.2 deal with the minimization of the restricted functional  $\overline{\mathcal{M}}_{ab}$ . Then, proposition 5.3 establishes the connection between the minimization of  $\overline{\mathcal{M}}_{ab}$  (or, equivalently, the minimization of  $\mathcal{M}_{ab}$ ) and the minimization of the original functional  $\mathcal{M}$  that ultimately leads to (5.10) and proposition 4.3. The angles  $\theta_a, \theta_b \in (0, \pi)$  defining  $\overline{\mathcal{M}}_{ab}$  are arbitrary.

**Proposition 5.1.** *For any critical point  $\bar{\gamma}_c$  of  $\overline{\mathcal{M}}_{ab}$  there are constants  $\epsilon > 0$  and  $c > 0$ , such that if  $\|\bar{\gamma} - \bar{\gamma}_c\|_{1,2} \leq \epsilon$  then*

$$\overline{\mathcal{M}}_{ab}(\bar{\gamma}) \geq \overline{\mathcal{M}}_{ab}(\bar{\gamma}_c) + c \|\bar{\gamma} - \bar{\gamma}_c\|_{1,2}^2. \tag{5.24}$$

*In particular  $\overline{\mathcal{M}}_{ab}$  achieves a strict local minimum at any of its critical points.*

**Proposition 5.2.**  *$\overline{\mathcal{M}}_{ab}$  has only one critical point  $\bar{\gamma}_c = \bar{\gamma}_e$  and  $\overline{\mathcal{M}}_{ab}(\bar{\gamma}_e)$  is a global minimum, i.e.*

$$\overline{\mathcal{M}}_{ab}(\bar{\gamma}) \geq \overline{\mathcal{M}}_{ab}(\bar{\gamma}_e). \tag{5.25}$$

**Proposition 5.3.** *We have*

$$\mathcal{M}_{ab}(\gamma, \hat{A}, \hat{a}) \geq \mathcal{M}_{ab}(\gamma_e, \hat{A}, \hat{a}) \tag{5.26}$$

*for functions  $\gamma = (\sigma, \omega)$  having the boundary values  $\gamma|_{\theta_a, \theta_b} = \gamma_e|_{\theta_a, \theta_b}$ .*

*Moreover, taking the limit  $[\theta_a, \theta_b] \rightarrow [0, \pi]$  we have*

$$\mathcal{M}(\gamma, \hat{A}, \hat{a}) \geq \mathcal{M}(\gamma_e, \hat{A}, \hat{a}). \tag{5.27}$$

*The explicit form of  $\mathcal{M}(\gamma_e)$  gives (5.10).*

Note that taking the limit  $(\theta_a, \theta_b) \rightarrow (0, \pi)$  is a very delicate issue as the limit boundary values of  $\sigma$  are not necessarily the same as those of  $\sigma_e$ . We will treat this problem following the ideas of [1].

**Proof of proposition 5.1.** For given  $\bar{\gamma}$  let  $\tilde{\gamma} = (\tilde{\sigma}, \tilde{\omega}) := \bar{\gamma} - \bar{\gamma}_c$  and define the path  $\bar{\gamma}_\tau = \bar{\gamma}_c + \tau\tilde{\gamma}$  for  $\tau$  in  $[0, 1]$ . The Taylor expansion of  $\overline{\mathcal{M}}_{ab}(\bar{\gamma}_\tau)$  at  $\tau = 0$  gives

$$\overline{\mathcal{M}}_{ab}(\bar{\gamma}) = \overline{\mathcal{M}}_{ab}(\bar{\gamma}_c) + \frac{1}{2} \partial_\tau^2 \overline{\mathcal{M}}_{ab}(\bar{\gamma}_\tau) \Big|_{\tau=0} + \frac{1}{6} \partial_\tau^3 \overline{\mathcal{M}}_{ab}(\bar{\gamma}_\tau) \Big|_{\tau=\tau^*}, \tag{5.28}$$

where  $0 \leq \tau^* \leq 1$ . The proof of proposition (5.1) comes from analysing the last two terms on the right hand side of (5.28). We do that separately.

To simplify notation set  $\overline{\mathcal{M}}_{ab}(\bar{\gamma}_\tau) = \overline{\mathcal{M}}_{ab}$ . Moreover, in the present proof primes on functions denote derivatives  $\partial_{\bar{\theta}}$ . The first  $\tau$ -derivative of  $\overline{\mathcal{M}}_{ab}$  as a function of  $\tau$  is



$$\partial_\tau \overline{\mathcal{M}}_{ab} = 2 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left[ \widehat{D} \bar{\sigma} \cdot \widehat{D} \bar{\sigma} + 2\bar{\sigma} + \frac{\widehat{D} \bar{\omega} \cdot \widehat{D} \bar{\omega} - \bar{\sigma} (\widehat{D} \bar{\omega})^2}{\bar{\eta}^2} - \frac{\bar{\sigma} \mathcal{V} e^{-\bar{\sigma}}}{2} \right] \sin \bar{\theta} d\bar{\theta}, \quad (5.29)$$

where

$$\mathcal{V} := 4\Lambda \left( \frac{\hat{A}}{4\pi} \right)^2 \frac{\sin^4 \theta}{\sin^4 \bar{\theta}} \hat{\chi}^2(\theta) \quad (5.30)$$

and the derivative operator  $\widehat{D}$  and the dot products are taken with respect to the standard metric on  $\mathbb{S}^2$ . (Due to axisymmetry  $\widehat{D} = \partial_{\bar{\theta}}$ ). Evaluate at  $\tau = 0$ , integrate by parts and use the boundary conditions to obtain the Euler–Lagrange equations for  $\overline{\mathcal{M}}_{ab}$ , namely

$$\widehat{\Delta} \bar{\sigma}_c - 2 + \frac{(\widehat{D} \bar{\omega}_c)^2}{\bar{\eta}_c^2} = -\frac{\mathcal{V}}{2} e^{-\bar{\sigma}_c}, \quad (5.31)$$

$$\widehat{D} \left( \frac{\widehat{D} \bar{\omega}_c}{\bar{\eta}_c^2} \right) = 0, \quad (5.32)$$

where  $\bar{\eta}_c = e^{\bar{\sigma}_c} \sin^2 \bar{\theta}$ , and  $\widehat{\Delta}$  is the Laplace operator with respect to the standard metric on  $\mathbb{S}^2$ . (Again, due to axisymmetry,  $\widehat{\Delta}$  involves only derivatives with respect to  $\bar{\theta}$ ).

The second  $\tau$ -derivative of  $\overline{\mathcal{M}}_{ab}$  reads

$$\begin{aligned} \partial_\tau^2 \overline{\mathcal{M}}_{ab} \\ = 2 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left[ (\widehat{D} \bar{\sigma})^2 + \frac{2\bar{\sigma}^2 (\widehat{D} \bar{\omega})^2 - 4\bar{\sigma} \widehat{D} \bar{\omega} \cdot \widehat{D} \bar{\omega} + (\widehat{D} \bar{\omega})^2}{\bar{\eta}^2} + \frac{\bar{\sigma}^2 \mathcal{V} e^{-\bar{\sigma}}}{2} \right] \sin \bar{\theta} d\bar{\theta}. \end{aligned} \quad (5.33)$$

Next, recall Carter's identity in the form (see [8])

$$F + \bar{\sigma} G'_\sigma + \bar{\omega} G'_\omega + 2\bar{\sigma} \bar{\omega} G_\omega - \bar{\eta}^{-2} \bar{\omega}^2 G_\sigma = H, \quad (5.34)$$

where

$$G_\sigma(\tau) = \widehat{\Delta} \bar{\sigma} + \bar{\eta}^{-2} (\widehat{D} \bar{\omega})^2 - 2, \quad (5.35)$$

$$G_\omega(\tau) = \widehat{D} (\bar{\eta}^{-2} \widehat{D} \bar{\omega}), \quad (5.36)$$

$$G'_\sigma(\tau) = \widehat{\Delta} \bar{\sigma} + \bar{\eta}^{-2} (2\widehat{D} \bar{\omega} \cdot \widehat{D} \bar{\omega} - 2\bar{\sigma} (\widehat{D} \bar{\omega})^2), \quad (5.37)$$

$$G'_\omega(\tau) = \widehat{D} (\bar{\eta}^{-2} (\widehat{D} \bar{\omega} - 2\bar{\sigma} \widehat{D} \bar{\omega})), \quad (5.38)$$

and

$$F(\tau) = (\widehat{D} \bar{\sigma} + \bar{\omega} \bar{\eta}^{-2} \widehat{D} \bar{\omega})^2 + (\widehat{D} (\bar{\omega} \bar{\eta}^{-1} - \bar{\eta}^{-1} \bar{\sigma} \widehat{D} \bar{\omega}))^2 + (\bar{\eta}^{-1} \bar{\sigma} \widehat{D} \bar{\omega} - \bar{\omega} \bar{\eta}^{-2} \widehat{D} \bar{\omega})^2, \quad (5.39)$$

$$H(\tau) = \widehat{D} (\bar{\sigma} \widehat{D} \bar{\sigma} + \bar{\omega} \bar{\eta}^{-1} \widehat{D} (\bar{\omega} \bar{\eta}^{-1})). \quad (5.40)$$

Now we can use the expressions for  $G'_\sigma$  and  $G'_\omega$  to obtain, after a simple integration by parts,

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} = -2 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( \bar{\sigma} G'_\sigma + \bar{\omega} G'_\omega - \frac{1}{2} \bar{\sigma}^2 \mathcal{V} e^{-\bar{\sigma}} \right) \sin \bar{\theta} d\bar{\theta}. \quad (5.41)$$

Using (5.34), integrating by parts once again and using the boundary conditions  $\bar{\sigma}(\bar{\theta}_a) = \bar{\sigma}(\bar{\theta}_b) = 0$ ,  $\bar{\omega}(\bar{\theta}_a) = \bar{\omega}(\bar{\theta}_b) = 0$  to get rid of  $H$ , yields

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} = 2 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( F + 2\bar{\sigma}\bar{\omega}G_\omega - \bar{\eta}^{-2}\bar{\omega}^2G_\sigma + \frac{1}{2}\bar{\sigma}^2\mathcal{V} e^{-\bar{\sigma}} \right) \sin \bar{\theta} d\bar{\theta}. \quad (5.42)$$

Evaluating at  $\tau = 0$  and using the Euler–Lagrange equations, we obtain

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} = 2 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( F + \frac{1}{2}(\bar{\eta}_c^{-2}\bar{\omega}^2 + \bar{\sigma}^2)\mathcal{V} e^{-\bar{\sigma}_c} \right) \sin \bar{\theta} d\bar{\theta}, \quad (5.43)$$

which can be written in the form

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} = 2 \int_{\theta_a}^{\theta_b} \left\{ \left( \bar{\sigma}' + \left( \frac{\omega'_c}{\bar{\eta}_c} \right) \left[ \frac{\bar{\omega}}{\bar{\eta}_c} \right] \right)^2 + \left( \left[ \frac{\bar{\omega}}{\bar{\eta}_c} \right]' - \left( \frac{\omega'_c}{\bar{\eta}_c} \right) \bar{\sigma} \right)^2 \right. \quad (5.44)$$

$$\left. + \left( \frac{\bar{\sigma}\omega'_c}{\bar{\eta}_c} - \frac{\bar{\omega}\bar{\eta}'_c}{\bar{\eta}_c^2} \right)^2 + \frac{\mathcal{V}}{2} \left( \left[ \frac{\bar{\omega}}{\bar{\eta}_c} \right]^2 + \bar{\sigma}^2 \right) e^{-\bar{\sigma}_c} \right\} \sin \bar{\theta} d\bar{\theta}. \quad (5.45)$$

We proceed by taking advantage of this formula.

First we note that because  $\bar{\gamma}_c$  is a critical point we have  $\omega'_c/\bar{\eta}_c^2 = k/\sin \bar{\theta}$  where  $k$  is a constant. Write  $\tilde{\omega} := \bar{\omega}/\bar{\eta}_c$  and disregard the first term in (5.45). We get

$$\begin{aligned} \partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq & 2 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left\{ \left( \bar{\sigma}' + \left( \frac{k\bar{\eta}_c}{\sin \bar{\theta}} \right) \tilde{\omega} \right)^2 + \left( \tilde{\omega}' - \left( \frac{k\bar{\eta}_c}{\sin \bar{\theta}} \right) \bar{\sigma} \right)^2 \right. \\ & \left. + \frac{\mathcal{V}}{2} (\tilde{\omega}^2 + \bar{\sigma}^2) e^{-\bar{\sigma}_c} \right\} \sin \bar{\theta} d\bar{\theta}. \end{aligned} \quad (5.46)$$

Let  $s := \min \{ (\sin \bar{\theta})/\bar{\eta}_c \}$  and assume

$$\int_\Omega \bar{\sigma}'^2 \sin \bar{\theta} d\bar{\theta} > \frac{4k^2}{s^2} \int_\Omega \tilde{\omega}^2 \sin \bar{\theta} d\bar{\theta}. \quad (5.47)$$

Then the first term in (5.46) can be bounded as

$$\left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( \bar{\sigma}' + \left( \frac{k\bar{\eta}_c}{\sin \bar{\theta}} \right) \tilde{\omega} \right)^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} \geq \quad (5.48)$$

$$\geq \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \bar{\sigma}'^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} - \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( \frac{k^2\bar{\eta}_c^2}{\sin^2 \bar{\theta}} \right) \tilde{\omega}^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} \quad (5.49)$$

$$\geq \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \bar{\sigma}'^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} - \frac{|k|}{s} \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \bar{\omega}^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} \quad (5.50)$$

$$\geq \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \bar{\sigma}'^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} - \frac{1}{2} \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \bar{\sigma}'^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} \quad (5.51)$$

$$= \frac{1}{2} \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \bar{\sigma}'^2 \sin \bar{\theta} d\bar{\theta} \right]^{1/2} \geq \min \left\{ \frac{\sin^{1/2} \bar{\theta}}{2} \right\} \left[ \int_{\bar{\theta}_a}^{\bar{\theta}_b} \bar{\sigma}'^2 d\bar{\theta} \right]^{1/2}, \quad (5.52)$$

where (5.51) has been obtained using (5.47). This bound together with the last term in (5.46) gives us

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq c_1 \|\bar{\sigma}\|_{1,2}^2 \quad (5.53)$$

for some constant  $c_1 > 0$ .

Now assume that the opposite to (5.47) holds, namely

$$\int_{\Omega} \bar{\sigma}'^2 \sin \bar{\theta} d\bar{\theta} \leq \frac{4k^2}{s^2} \int_{\Omega} \bar{\omega}^2 \sin \bar{\theta} d\bar{\theta}. \quad (5.54)$$

Then from (5.46) we have

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq \int_{\Omega} \mathcal{V} (\bar{\omega}^2 + \bar{\sigma}^2) e^{-\bar{\sigma}_c} \sin \bar{\theta} d\bar{\theta} \quad (5.55)$$

$$\geq \min \{ \mathcal{V} e^{-\bar{\sigma}_c} \} \int_{\bar{\theta}_a}^{\bar{\theta}_b} (\bar{\omega}^2 + \bar{\sigma}^2) \sin \bar{\theta} d\bar{\theta} \quad (5.56)$$

$$\geq \min \{ \mathcal{V} e^{-\bar{\sigma}_c} \} \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( \frac{s^2}{4k^2} \bar{\sigma}'^2 + \bar{\sigma}^2 \right) \sin \bar{\theta} d\bar{\theta} \quad (5.57)$$

$$\geq \min \{ \mathcal{V} e^{-\bar{\sigma}_c} \} \min \left\{ 1, \frac{s^2}{4k^2} \right\} \int_{\bar{\theta}_a}^{\bar{\theta}_b} (\bar{\sigma}'^2 + \bar{\sigma}^2) \sin \bar{\theta} d\bar{\theta} \quad (5.58)$$

which again gives us an inequality  $\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq c_2 \|\bar{\sigma}\|_{1,2}^2$  for some constant  $c_2 > 0$ . Thus in either case we have

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq c_3 \|\bar{\sigma}\|_{1,2}^2 \quad (5.59)$$

for some constant  $c_3 > 0$ .

Now we can interchange the roles of  $\bar{\sigma}$  and  $\bar{\omega}$  (observing the symmetry in (5.46)) to find again

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq c_3 \|\bar{\omega}\|_{1,2}^2. \quad (5.60)$$

Using that  $\bar{\omega} = \bar{\omega}/\eta_c$  and by an argument similar to the previous one we deduce from (5.60) that

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq c_4 \|\bar{\omega}\|_{1,2}^2 \quad (5.61)$$

for some constant  $c_4 > 0$ . Collecting (5.59) and (5.61) we get

$$\partial_\tau^2 \overline{\mathcal{M}}_{ab} |_{\tau=0} \geq c_5 \|(\bar{\sigma}, \bar{\omega})\|_{1,2}^2 = c_5 \|\bar{\gamma} - \bar{\gamma}_c\|_{1,2}^2 \quad (5.62)$$

for some constant  $c_5 > 0$ .

Having treated the second term on the right hand side of (5.28) we turn to the last one. We claim that there is a constant  $c_6 > 0$  such that if  $\|\bar{\gamma} - \bar{\gamma}_c\|_{1,2} \leq 1$  then

$$\partial_{\tau}^3 \overline{\mathcal{M}}_{ab} |_{\tau=\tau^*} \leq c_6 \|(\bar{\sigma}, \bar{\omega})\|_{1,2}^3. \quad (5.63)$$

Combined with (5.28) this would show, as we want, that if  $\|(\bar{\sigma}, \bar{\omega})\|_{1,2} \leq \epsilon$  for  $\epsilon$  sufficiently small, then (5.24) holds for some constant  $c > 0$ . The bound (5.63) is indeed easily obtained. A direct computation gives

$$\partial_{\tau}^3 \overline{\mathcal{M}}_{ab} = -2 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( \frac{6\bar{\sigma}\bar{\omega}'^2 - 12\bar{\sigma}^2\bar{\omega}'\bar{\omega}' + 4\bar{\sigma}^3\bar{\omega}'^2}{\bar{\eta}^2} + \frac{\mathcal{V}}{2} \bar{\sigma}^3 e^{-\bar{\sigma}} \right) \sin \bar{\theta} d\bar{\theta}. \quad (5.64)$$

Bounds for each term in this integral, compatible with (5.63), are obtained by using that  $\|\bar{\sigma}\|_{\infty} \leq \sqrt{\pi} \|\bar{\sigma}\|_{1,2} \leq \sqrt{\pi} \|(\bar{\sigma}, \bar{\omega})\|_{1,2}$ , and that if  $\|\bar{\gamma} - \bar{\gamma}_c\|_{1,2} \leq 1$  then  $\|\bar{\sigma}\|_{\infty} \leq c_7$  and  $\|\bar{\omega}\|_2 \leq c_8$  for constants  $c_7 > 0$  and  $c_8 > 0$ . For instance the first term is bounded as

$$\left| 12 \int_{\bar{\theta}_a}^{\bar{\theta}_b} \frac{\bar{\sigma}\bar{\omega}'^2}{\bar{\eta}^2} \sin \bar{\theta} d\bar{\theta} \right| \leq 12 \sup \left\{ \frac{1}{\sin^3 \theta} \right\} e^{2c_7} \|\bar{\sigma}\|_{\infty} \|\bar{\omega}\|_2^2 \leq c_9 \|(\bar{\sigma}, \bar{\omega})\|_{1,2}^3 \quad (5.65)$$

for some constant  $c_9 > 0$ . The other terms are bounded in the same way.  $\square$

**Proof of proposition 5.2.** It will be more convenient to work with the functional  $\mathcal{M}_{ab}^*(\gamma^*)$  of the arguments  $\gamma^* = (u, \omega)$  with  $u = -\ln \eta$ , given by

$$\mathcal{M}_{ab}^*(\gamma^*) = \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( u'^2 + \omega'^2 e^{2u} + \mathcal{V}^* e^u \right) \sin \bar{\theta} d\bar{\theta}, \quad (5.66)$$

where

$$\mathcal{V}^* = \mathcal{V} \sin^2 \bar{\theta}. \quad (5.67)$$

This functional is equal to  $\mathcal{M}_{ab}(\gamma)$  plus a constant independent of the arguments. (Use  $u = -\ln \eta$  in (5.23)).

If  $\mathcal{M}_{ab}^*$  is shown to satisfy the Palais–Smale (PS) condition (see below), then a simple application of proposition 5.1 and the mountain pass theorem, as explained in the *corollary* on page 187 of [15], shows that  $\gamma_e^* = (\ln \eta_e, \omega_e)$  is the only critical point and that  $\mathcal{M}_{ab}^*(\gamma_e^*)$  is the strict absolute minimum of  $\mathcal{M}_{ab}^*$ .

We explain now how to verify the PS condition. Recall first that the PS condition holds iff any sequence  $\gamma_i^*$  for which  $\mathcal{M}_{ab}^*(\gamma_i^*)$  is bounded and for which  $\|\delta \mathcal{M}_{ab}^*(\gamma_i^*)\| \rightarrow 0$  has a (strongly) convergent subsequence. Here  $\|\delta \mathcal{M}_{ab}^*(\gamma_i^*)\|$  is the norm of the differential of  $\mathcal{M}_{ab}^*$  at  $\gamma_i^*$ . Recall that this norm is  $\|\delta \mathcal{M}_{ab}^*(\gamma^*)\| = \sup \left\{ |\delta_X \mathcal{M}_{ab}^*(\gamma^*)| : \|X\|_{1,2} = 1 \right\}$ . Note from this definition that if  $\|\delta \mathcal{M}_{ab}^*(\gamma_i^*)\| \rightarrow 0$ , then for any sequence  $X_i$  with  $\|X_i\|_{1,2} \leq K$  we have

$$\left| \delta_{X_i} \mathcal{M}_{ab}^*(\gamma_i^*) \right| \rightarrow 0. \quad (5.68)$$

Now, for any tangent vector  $X = (\tilde{u}, \tilde{\omega})$  to a point  $\gamma^* = (u, \omega)$  we compute

$$\delta_X \mathcal{M}_{ab}^*(\gamma^*) = \int_{\bar{\theta}_a}^{\bar{\theta}_b} \left( 2\tilde{u}'u' + 2\tilde{u}\omega'^2 e^{2u} + 2\tilde{\omega}'\omega' e^{2u} + \tilde{u}\mathcal{V} \sin^2 \bar{\theta} e^u \right) \sin \bar{\theta} d\bar{\theta}. \quad (5.69)$$

This expression will be used below.

Let  $\gamma_i^*$  be a sequence such that  $\mathcal{M}_{ab}^*(\gamma_i^*)$  is uniformly bounded and such that  $\|\delta\mathcal{M}_{ab}^*(\gamma_i^*)\| \rightarrow 0$ . From (5.66) we deduce that  $\|u_i'\|_2$  is uniformly bounded<sup>4</sup> and from this and (5.17) that  $u_i$  is uniformly bounded and uniformly continuous. By the theorem of Arzelà – Ascoli,  $u_i$  has a  $C^0$ -convergent subsequence (that we still index by ‘ $i$ ’). As  $\|u_i\|_{1,2}$  is uniformly bounded we can assume that  $u_i$  converges weakly in  $H^{1,2}$  too. Then, from the  $C^0$ -boundedness of  $u_i$  and again from (5.66), we deduce in a similar fashion that  $\omega_i$  has a subsequence converging in  $C^0$  and weakly in  $H^{1,2}$ .

Assume then without loss of generality that for the above sequence  $\gamma_i^*$  we have  $u_i \rightarrow u_\infty$  and  $\omega_i \rightarrow \omega_\infty$  weakly in  $H^{1,2}$  and strongly in  $C^0$ . Let  $c > 0$  be a constant such that  $c < e^{2u_i} \sin \bar{\theta}$  for all  $i$ . Then,

$$\begin{aligned} c \int_{\bar{\theta}_a}^{\bar{\theta}_b} (\omega_i' - \omega_\infty')^2 d\bar{\theta} &\leq \int_{\bar{\theta}_a}^{\bar{\theta}_b} (\omega_i' - \omega_\infty')^2 e^{2u_i} \sin \bar{\theta} d\bar{\theta} \\ &= \left( \int_{\bar{\theta}_a}^{\bar{\theta}_b} \omega_i' (\omega_i' - \omega_\infty') e^{2u_i} \sin \bar{\theta} d\bar{\theta} - \int_{\bar{\theta}_a}^{\bar{\theta}_b} \omega_\infty' (\omega_i' - \omega_\infty') e^{2u_i} \sin \bar{\theta} d\bar{\theta} \right) \rightarrow 0, \end{aligned} \quad (5.70)$$

where the first integral in (5.70) is seen to go to zero by taking  $V_i = (0, \tilde{\omega}_i)$  with  $\tilde{\omega}_i = \omega_i - \omega_\infty$  in (5.68), while the second integral in (5.70) tends to zero because  $\omega_i \rightarrow \omega_\infty$  weakly in  $H^{1,2}$  and  $u_i \rightarrow u_\infty$  strongly in  $C^0$  and weakly in  $H^{1,2}$ . From (5.17) and (5.70) we deduce that  $\|\omega_i - \omega_\infty\|_2 \rightarrow 0$ , which together with (5.70) again shows that  $\omega_i \rightarrow \omega_\infty$  in  $H^{1,2}$ . The convergence  $u_i \rightarrow u_\infty$  in  $H^{1,2}$  is shown in the same fashion.

**Proof of proposition 5.3.** Inequality (5.26) follows from propositions 5.1 and 5.2, together with the relation (5.22) between the functionals  $\mathcal{M}_{ab}$  and  $\overline{\mathcal{M}}_{ab}$ , as they imply that extreme KdS data  $(\sigma_e, \omega_e)$  are the unique global minimizers of  $\mathcal{M}_{ab}$  among functions  $(\sigma, \omega)$  having the same boundary conditions as  $(\sigma_e, \omega_e)$  at  $\theta_a, \theta_b$ .

The proof of (5.27) is line by line identical to the proof when  $\Lambda = 0$  and which was obtained in [1]. We will only sketch the argument here and refer the reader to [1] for details. It is important to remark that the presence of the cosmological constant plays no important role in this step.

Divide the interval  $[0, \pi]$  in three regions,  $\Omega_I = \{\sin \theta \leq e^{(\ln t)^2}\}$ ,  $\Omega_{II} = \{e^{(\ln t)^2} \leq \sin \theta \leq t\}$  and  $\Omega_{III} = \{t \leq \sin \theta\}$ . Note that when  $t$  goes to zero, the regions  $\Omega_I$  and  $\Omega_{II}$  shrink toward the poles, while  $\Omega_{III}$  extends to cover the whole interval  $[0, \pi]$ . Then a specific partition function  $f(\theta)$  (see equations (70)-(71) in [1]) is used to interpolate between extreme KdS horizon data in region  $\Omega_I$  and general data in region  $\Omega_{III}$ . Define the auxiliary interpolating data  $\gamma(t) = (\sigma(t), \omega(t))$  as

$$\gamma(t) = f_t(\sin \theta)\gamma + (1 - f_t(\sin \theta))\gamma_e, \quad (5.71)$$

then, as mentioned before, combining propositions 5.1 and 5.2 on the region  $[\theta_a, \theta_b] := \Omega_{II} \cup \Omega_{III}$  for functions  $\gamma(t): \Gamma_{ab} \rightarrow \mathbb{R}^2$  we find

$$\mathcal{M}_{ab}(\gamma(t)) \geq \mathcal{M}_{ab}(\gamma_e). \quad (5.72)$$

Moreover, as  $\gamma(t)|_{\Omega_I} = \gamma_e|_{\Omega_I}$ , we can extend (5.72) to  $[0, \pi]$  (recall that  $[0, \pi] = \Omega_I \cup [\theta_a, \theta_b]$ ) to obtain

<sup>4</sup> Note that there are constants  $0 < c_1 < c_2 < \infty$  such that  $c_1 < \sin \theta < c_2$ .

$$\mathcal{M}(\gamma(t)) \geq \mathcal{M}(\gamma_e). \quad (5.73)$$

The final step is to show that as  $t$  goes to zero, the mass functional for the auxiliary data converges to the mass functional for the original general data, that is

$$\lim_{t \rightarrow 0} \mathcal{M}(\gamma(t)) = \mathcal{M}(\gamma). \quad (5.74)$$

This is done in an identical manner as in [1] (with  $\Lambda$  being irrelevant here), by using that  $\omega = \omega_e + \mathcal{O}(\sin^2 \theta)$  near the poles and that  $\mathcal{M}(\gamma)$  and  $\mathcal{M}(\gamma_e)$  are well defined.

Inequalities (5.73) and (5.74) give (5.27).

Moreover, using the explicit value

$$e^{\frac{\mathcal{M}(\sigma_e, \omega_e, \hat{A}, \hat{a}) - \hat{\beta}}{8\hat{\kappa}}} = \frac{\hat{A}}{4\pi} \quad (5.75)$$

we find

$$e^{\frac{\mathcal{M}(\sigma, \omega, \hat{A}, \hat{a}) - \hat{\beta}}{8\hat{\kappa}}} \geq \frac{\hat{A}}{4\pi} \quad (5.76)$$

which is inequality (5.10).  $\square$

## 6. Possible generalizations

We conclude discussing possible extensions of our main result to the case with electromagnetic field and to the case  $\Lambda < 0$ . In the former case we conjecture an inequality which, in addition to  $A$ ,  $J$  and  $\Lambda$ , contains electric and magnetic charges  $Q_E$  and  $Q_M$  in the combination  $Q^2 = Q_E^2 + Q_M^2$ . Such an extension is natural from the fact that all special cases are proven, in particular we recall [6] the bound  $A^2 \geq 16\pi^2(4J^2 + Q^4)$  in the case  $\Lambda = 0$ . Moreover, extreme Kerr–Newman–de Sitter saturates (6.1) and (6.2).

**Conjecture 6.1.** Under the assumptions of theorem 1.1 but under the presence of an electromagnetic field with charges  $Q_E$ ,  $Q_M$  with  $Q^2 = Q_E^2 + Q_M^2$  and for any  $\Lambda > 0$  we have

$$J^2 \leq \frac{A^2}{64\pi^2} \left[ \left(1 - \frac{\Lambda A}{4\pi}\right) \left(1 - \frac{\Lambda A}{12\pi}\right) - \frac{2\Lambda Q^2}{3} \right] - \frac{Q^4}{4} \quad (6.1)$$

or equivalently,

$$\left( Q^2 + \frac{\Lambda A^2}{48\pi^2} - \sqrt{\frac{A^2}{16\pi^2} \left(1 - \frac{\Lambda A}{6\pi}\right)^2 - 4J^2} \right) \times \left( Q^2 + \frac{\Lambda A^2}{48\pi^2} + \sqrt{\frac{A^2}{16\pi^2} \left(1 - \frac{\Lambda A}{6\pi}\right)^2 - 4J^2} \right) \leq 0. \quad (6.2)$$

Moreover, (6.1) and (6.2) are saturated precisely for extreme Kerr–Newman–de Sitter configurations.

As to the calculations leading to (6.1) and (6.2) we made use of equation (44) of Caldarelli *et al* [4], where the temperature  $T$  of a Kerr–Newman–anti-de Sitter black hole is

given in terms of  $l^2 = -3/\Lambda$ , the mass  $M$ , the entropy  $S = A/4$ ,  $Q$  and  $J$ . This calculation is insensitive to the sign of  $\Lambda$ , and the requirement that  $T \geq 0$  gives directly (6.1), while (6.2) is obtained via simple algebraic manipulations.

We finally comment on the prospects of proving the area inequalities (1.5), (6.1) and (6.2) for the case  $\Lambda < 0$  along the lines described above. We first remark that extreme Kerr–anti-de Sitter saturates (1.5) which should be clear from the discussion of section 3, and extreme Kerr–Newman–anti-de Sitter saturates (6.1) and (6.2). Next, the first part of our proof of (1.5), namely the lower bound for  $A$  in terms on  $\mathcal{M}$  as given in (4.8) carries over to  $\Lambda < 0$  straightforwardly. However, attempts of obtaining a lower bound for  $\mathcal{M}$  analogously to (4.12) seem to be in vain. The reason is that one can easily construct examples with sufficiently small  $\sigma$  (negative with large modulus), and suitably adjusted  $\omega$  for which the last term in (4.9), which is now negative, dominates the first two positive terms. In fact these examples strongly suggest that  $\mathcal{M}$  is even unbounded from below unless the data are restricted appropriately. Therefore, while it is still possible that (1.5), (6.1) and (6.2) hold for  $\Lambda < 0$  as well, our strategy which was successful for  $\Lambda > 0$  is unlikely to carry over.

## Acknowledgments

We acknowledge helpful discussions with Lars Andersson, Piotr Bizoń, Piotr Chruściel, Sergio Dain, Jose Luis Jaramillo, Marc Mars and Luc Nguyen. MEGC is supported by CONICET (Argentina). WS was funded by the Austrian Science Fund (FWF): P23337-N16 and by the Albert Einstein Institute (Potsdam).

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